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Noncommutative Monopole from Nonlinear Monopole

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Abstract

We solve the non-linear monopole equation of the Born-Infeld theory to all orders in the NS 2-form and give physical implications of the result. The solution is constructed by extending the earlier idea of rotating the brane configuration of the Dirac monopole in the target space. After establishing the non-linear monopole, we explore the non-commutative monopole by the Seiberg-Witten map.

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1 Introduction

Recently non-commutative gauge theory has received much attention for its origin in string theory. The effective action of D-brane in string theory with a constant NS 2-form B_{ij} is the non-commutative Born-Infeld theory when the point splitting regularization is adopted [1, 2]. On the other hand, if we adopt the Pauli-Villars regularization we obtain the ordinary Born-Infeld theory. Since the method of regularization should not change the physical S-matrices, the two descriptions are argued to be related by a field redefinition [3] (called the Seiberg-Witten map).

This relation has been investigated intensively from various aspects, and among other things, from various BPS solutions [4, 5, 6, 7, 8, 9]. Since the constant NS 2-form serves as a uniform magnetic field, if we view the monopole solution as a D-string ending on a D3-brane, we expect the D-string tilts due to force balance between the D-string tension and the magnetic force at the endpoint [4]. This system was analyzed in [10] as the solution of the linearly realized BPS equation in the commutative space.

However if we would like to see the tilts directly from the non-commutative viewpoint, it would be a hard task. We can only solve the linearly realized BPS equation in the perturbation expansion with respect to the non-commutativity parameter θ. And even if we have solved the BPS equation, we would have to know how to extract the eigenvalues for the brane interpretation of [11, 12]. In our previous works [5, 7] we proposed the non-commutative eigenvalue equation to analyze the asymptotic behavior and confirmed the tilts.

In [8, 9], it was discussed that we can give the brane interpretation by transforming into the commutative viewpoint by the Seiberg-Witten map. Actually they argued that the linearly realized BPS equation in the non-commutative space is mapped to the non-linearly realized BPS equation in the commutative space by extending the instanton case [3]. Moreover the latter is related to the linearly realized BPS equation in the commutative space by the rotation in the target space. In the electric case of [8], an exact treatment of the soliton was given though there are no discussions on the non-linearly realized BPS equation. In the magnetic case of [9] the discussion on the non-linearly realized BPS equation was restricted to the approximation $r^2 \gg 2\pi\alpha' \gg (2\pi\alpha')^2 B$.

In this paper we shall extend the works of [8, 9] to explore the non-commutative BPS soliton from the non-linear BPS soliton in the commutative space. First we shall solve the non-linearly realized BPS equation exactly without any approximation. We find that the solution is nothing but the one obtained by rotating the solution of the linearly realized BPS equation in the target space.

After establishing the soliton of the non-linearly realized BPS equation in the commutative space, we explicitly write down the first few terms in the expansion of the NS 2-form B_{ij}. What we find is terms in a mess and at first sight it seems impossible that they are related to the non-commutative monopole by the Seiberg-Witten map. We shall resolve this problem by using the moduli of the open string, namely, the open string metric G and the non-commutativity parameter θ. Finally we map the non-linear monopole into the non-commutative space. We confirm that it satisfies the non-commutative BPS equation up to $O(\theta^2)$.

2 Nonlinear BPS equation

In this section, we shall explicitly solve the non-linear BPS equation in the commutative space. First we shall recall the linearly realized supersymmetries δ_{L} and non-linearly realized supersymmetries δ_{NL} [13, 14, 15, 3] of the Born-Infeld action:

$$\delta_{\mathrm{L}}\lambda = \frac{1}{2\pi\alpha'}M^{+}_{mn}\sigma^{mn}\eta, \tag{1}$$

$$\delta_{\mathrm{L}}\bar{\lambda} = \frac{1}{2\pi\alpha'}M^{-}_{mn}\sigma^{mn}\bar{\eta}, \tag{2}$$

$$\delta_{\mathrm{NL}}\lambda = \frac{1}{4\pi\alpha'}\Big(1 - \mathrm{Pf}\, M + \sqrt{1 - \mathrm{Tr}\, M^2/2 + (\mathrm{Pf}\, M)^2}\Big)\eta^*, \tag{3}$$

$$\delta_{\mathrm{NL}}\bar{\lambda} = \frac{1}{4\pi\alpha'}\Big(1 + \mathrm{Pf}\, M + \sqrt{1 - \mathrm{Tr}\, M^2/2 + (\mathrm{Pf}\, M)^2}\Big)\bar{\eta}^*, \tag{4}$$

where M denotes

$$M = (2\pi\alpha')\begin{pmatrix} 0 & -\partial_1\Phi & -\partial_2\Phi & -\partial_3\Phi \\ \partial_1\Phi & 0 & (F_3+B_3) & -(F_2+B_2) \\ \partial_2\Phi & -(F_3+B_3) & 0 & (F_1+B_1) \\ \partial_3\Phi & (F_2+B_2) & -(F_1+B_1) & 0 \end{pmatrix}, \tag{5}$$

with the magnetic field $F_i = \epsilon_{ijk}F_{jk}/2$, a constant NS 2-form background $B_i = \epsilon_{ijk}B_{jk}/2$ and the Higgs field Φ. Here we turn on only the spatial components of the field strength and the NS 2-form. The matrix M has been obtained by regarding the Euclidean time component of the gauge field as the Higgs field Φ and discarding the time derivatives. We shall set $2\pi\alpha' = 1$ for simplicity hereafter, however we can restore it on the dimensional ground anytime we like. The non-linear BPS equation [3, 16] is the condition for preserving the linear combination of δ_{L} and δ_{NL} which is unbroken at the infinity where the field strength and the derivative of the Higgs field vanish:

$$\frac{\boldsymbol{F}+\boldsymbol{B}-\boldsymbol{\partial}\Phi}{1+(\boldsymbol{F}+\boldsymbol{B})\cdot\boldsymbol{\partial}\Phi+\sqrt{1+(\boldsymbol{F}+\boldsymbol{B})^2+(\boldsymbol{\partial}\Phi)^2+\Big((\boldsymbol{F}+\boldsymbol{B})\cdot\boldsymbol{\partial}\Phi\Big)^2}} = \frac{\boldsymbol{B}}{1+\sqrt{1+\boldsymbol{B}^2}}. \tag{6}$$

This BPS equation is not so complicated to solve as it looks. The starting point is similar to the case of instanton [17]. First we note that eq. (6) implies $\boldsymbol{F}-\boldsymbol{\partial}\Phi$ is proportional to $\boldsymbol{B}$:

$$\boldsymbol{F}-\boldsymbol{\partial}\Phi = f\boldsymbol{B}, \tag{7}$$

where f is an unknown function. The key point to solve this BPS equation is to rewrite eq. (6) as

$$(f+1)\Big(1+\sqrt{1+\boldsymbol{B}^2}\Big)-1-(\boldsymbol{F}+\boldsymbol{B})\cdot\boldsymbol{\partial}\Phi=\sqrt{1+(\boldsymbol{F}+\boldsymbol{B})^2+(\boldsymbol{\partial}\Phi)^2+\Big((\boldsymbol{F}+\boldsymbol{B})\cdot\boldsymbol{\partial}\Phi\Big)^2}. \tag{8}$$

Taking the square of this equation (8) and using the relation (7) to eliminate the magnetic field $\boldsymbol{F}$ when necessary, we find that eq. (8) is reduced simply to

$$f=(\boldsymbol{\partial}\Phi)^2+(f+1)\boldsymbol{B}\cdot\boldsymbol{\partial}\Phi. \tag{9}$$

Another equation for f and Φ besides (9) is obtained by taking the divergence of the relation (7) and using the Bianchi identity $\boldsymbol{\partial}\cdot\boldsymbol{F}=0$,

$$-\boldsymbol{\partial}^2\Phi=\boldsymbol{B}\cdot\boldsymbol{\partial}f. \tag{10}$$

Now we have a system of differential equations (9) and (10) for the scalar quantities f and Φ. After eliminating f we find quite a non-linear equation for Φ:

$$\boldsymbol{\partial}^2\Phi\Big(1-\boldsymbol{B}\cdot\boldsymbol{\partial}\Phi\Big)^2+2\boldsymbol{B}\cdot\boldsymbol{\partial}\boldsymbol{\partial}\Phi\cdot\boldsymbol{\partial}\Phi\Big(1-\boldsymbol{B}\cdot\boldsymbol{\partial}\Phi\Big)+\boldsymbol{B}\cdot\boldsymbol{\partial}\boldsymbol{B}\cdot\boldsymbol{\partial}\Phi\Big(1+(\boldsymbol{\partial}\Phi)^2\Big)=0. \tag{11}$$

Hereafter we shall suppose the constant background $\boldsymbol{B}$ is in the z direction and rewrite the equation (11) in the cylindrical coordinate (ρ,φ,z) with $x=\rho\cos\varphi$ and $y=\rho\sin\varphi$,

$$\begin{aligned}(\partial_\rho^2\Phi+\partial_\rho\Phi/\rho+\partial_z^2\Phi)(1-B\partial_z\Phi)^2+2B(\partial_\rho\partial_z\Phi\partial_\rho\Phi+\partial_z^2\Phi\partial_z\Phi)(1-B\partial_z\Phi)&\\ +B^2\partial_z^2\Phi(1+(\partial_\rho\Phi)^2+(\partial_z\Phi)^2)=0.&\end{aligned} \tag{12}$$

This differential equation looks impossible to solve. However, we can apply the idea of [8, 9] to find that the solution is exactly the one obtained by rotating the solution of the linear BPS equation in the target space by an angle ϕ with $\tan\phi=B$. This idea is convincing for the following reason. Originally the string theory has the $SO(1,9)$ Lorentz symmetry and 32 supersymmetries. Taking the static gauge the Lorentz symmetry is broken into $SO(1,3)\times SO(6)$ and half of the supersymmetries are broken. The broken symmetries are realized non-linearly. If we rotate the target space and still take the static gauge by adopting a different worldsheet coordinate, we would find that originally linearly realized symmetries correspond in general to some combinations of the linear and the non-linear ones. Therefore the linear BPS equation and the non-linear BPS equation should be related by a target space rotation.

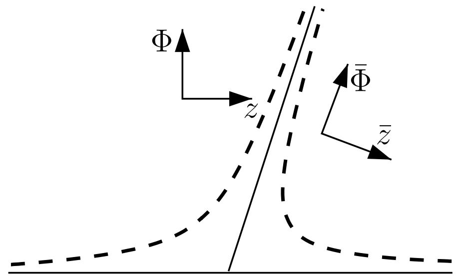


Figure 1: It is easier to find the solution using the coordinate $(\bar{\Phi}, \bar{z})$ instead of (Φ, z). The dashed line denotes the solution of the Higgs field to be found.

To see this explicitly we change our variables into those with bars by the target space rotation,

$$\begin{pmatrix} \bar{\Phi} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \Phi \\ z \end{pmatrix}, \tag{13}$$

and show that the solution for the variables with bars is the same as that of the linear BPS equation. First we have to rewrite the equation (12) by changing Φ, ∂_ρ, ∂_z into $\bar{\Phi}$, $\bar{\partial}_\rho \equiv \partial/\partial\rho|_{\bar{z}}$, $\bar{\partial}_z \equiv \partial/\partial\bar{z}|_\rho$. Note that, though we do not change the coordinate ρ, $\bar{\partial}_\rho$ is different form ∂_ρ because the coordinate to be fixed is different between them. The formulas for rewriting $\partial_z\Phi$ and $\partial_\rho\Phi$ into $\bar{\partial}_z\bar{\Phi}$ and $\bar{\partial}_\rho\bar{\Phi}$ read

$$\partial_z\Phi = \frac{\cos\phi\bar{\partial}_z\bar{\Phi} - \sin\phi}{\cos\phi + \sin\phi\bar{\partial}_z\bar{\Phi}}, \tag{14}$$

$$\partial_\rho\Phi = \frac{\bar{\partial}_\rho\bar{\Phi}}{\cos\phi + \sin\phi\bar{\partial}_z\bar{\Phi}}, \tag{15}$$

where the first formula is directly obtained from the rotation (13) and the second formula is a consequence of the chain rule formula

$$\left.\frac{\partial\bar{\Phi}}{\partial\rho}\right|_z = \left.\frac{\partial\rho}{\partial\rho}\right|_z \left.\frac{\partial\bar{\Phi}}{\partial\rho}\right|_{\bar{z}} + \left.\frac{\partial\bar{z}}{\partial\rho}\right|_z \left.\frac{\partial\bar{\Phi}}{\partial\bar{z}}\right|_\rho, \tag{16}$$

and the relations $\partial\bar{\Phi}/\partial\rho|_z = \cos\phi\,\partial\Phi/\partial\rho|_z$ and $\partial\bar{z}/\partial\rho|_z = -\sin\phi\,\partial\Phi/\partial\rho|_z$. In the same way we also find the similar formulas for higher derivatives,

$$\partial_z^2\Phi = \bar{\partial}_z^2\bar{\Phi}/(\cos\phi + \sin\phi\bar{\partial}_z\bar{\Phi})^3, \tag{17}$$

$$\partial_z\partial_\rho\Phi = \Big[\cos\phi\bar{\partial}_z\bar{\partial}_\rho\bar{\Phi} + \sin\phi(\bar{\partial}_z\bar{\partial}_\rho\bar{\Phi}\bar{\partial}_z\bar{\Phi} - \bar{\partial}_z^2\bar{\Phi}\bar{\partial}_\rho\bar{\Phi})\Big]/(\cos\phi + \sin\phi\bar{\partial}_z\bar{\Phi})^3, \tag{18}$$

$$\begin{aligned}\partial_\rho^2\Phi = {}& \Big[(\cos\phi)^2\bar{\partial}_\rho^2\bar{\Phi} + 2\cos\phi\sin\phi\Big(-\bar{\partial}_z\bar{\partial}_\rho\bar{\Phi}\bar{\partial}_\rho\bar{\Phi} + \bar{\partial}_\rho^2\bar{\Phi}\bar{\partial}_z\bar{\Phi}\Big) \\ & +(\sin\phi)^2\Big(\bar{\partial}_z^2\bar{\Phi}(\bar{\partial}_\rho\bar{\Phi})^2 - 2\bar{\partial}_z\bar{\partial}_\rho\bar{\Phi}\bar{\partial}_z\bar{\Phi}\bar{\partial}_\rho\bar{\Phi} + \bar{\partial}_\rho^2\bar{\Phi}(\bar{\partial}_z\bar{\Phi})^2\Big)\Big]/(\cos\phi + \sin\phi\bar{\partial}_z\bar{\Phi})^3.\end{aligned} \tag{19}$$

Using these formulas, the terribly non-linear equation (12) now becomes

$$\bar{\partial}_\rho^2 \bar{\Phi} + \bar{\partial}_\rho \bar{\Phi}/\rho + \bar{\partial}_z^2 \bar{\Phi} = 0, \tag{20}$$

which is nothing but the three-dimensional laplace equation. The solution to eq. (20) is given by the sum of the Coulomb term and the linear term determined by the boundary condition in the asymptotic region:

$$\bar{\Phi} = \frac{q}{\sqrt{\rho^2 + \bar{z}^2}} + B\bar{z}. \tag{21}$$

Turning back to the variables without bars using the relation (13), our final result for the Higgs field Φ is given as the solution of the algebraic equation,

$$\Big((1 + B^2)\rho^2 + z^2 - 2Bz\Phi + B^2\Phi^2\Big)\Phi^2 = q^2, \tag{22}$$

or its covariant form

$$\Big((1 + \boldsymbol{B}^2)\boldsymbol{x}^2 - (\boldsymbol{B}\cdot\boldsymbol{x})^2 - 2\boldsymbol{B}\cdot\boldsymbol{x}\Phi + \boldsymbol{B}^2\Phi^2\Big)\Phi^2 = q^2. \tag{23}$$

The explicit expression of the first few terms in the expansion with respect to B is

$$\Phi = \frac{q}{r} + \frac{q^2\boldsymbol{B}\cdot\boldsymbol{x}}{r^4} - \frac{1}{2}\frac{q\boldsymbol{B}^2}{r} + \frac{1}{2}\frac{q(\boldsymbol{B}\cdot\boldsymbol{x})^2}{r^3} - \frac{1}{2}\frac{q^3\boldsymbol{B}^2}{r^5} + \frac{5}{2}\frac{q^3(\boldsymbol{B}\cdot\boldsymbol{x})^2}{r^7}, \tag{24}$$

with $r = \sqrt{\rho^2 + z^2}$.

Similarly the magnetic field is also obtained from the relations (7) and (9) as

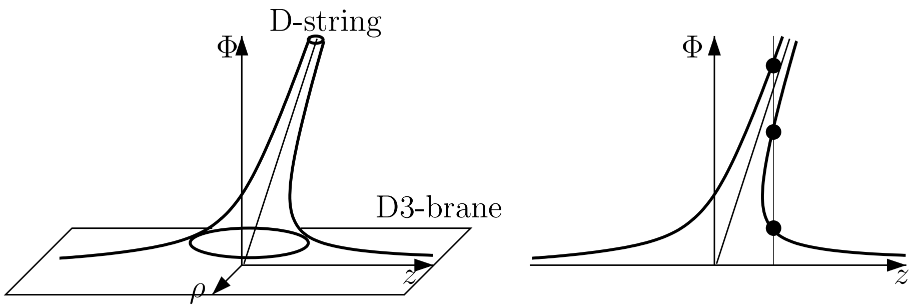
$$\boldsymbol{F} = \boldsymbol{\partial}\Phi + \frac{(\boldsymbol{\partial}\Phi)^2 + \boldsymbol{B}\cdot\boldsymbol{\partial}\Phi}{1 - \boldsymbol{B}\cdot\boldsymbol{\partial}\Phi}\boldsymbol{B}. \tag{25}$$

Using our result (23) we can rewrite this expression (25) by eliminating the derivatives of the Higgs field $\boldsymbol{\partial}\Phi$,

$$\boldsymbol{F} = \frac{(1 + \boldsymbol{B}^2)(-\boldsymbol{x} + 2\Phi\boldsymbol{B})\Phi}{(1 + \boldsymbol{B}^2)\boldsymbol{x}^2 - (\boldsymbol{B}\cdot\boldsymbol{x})^2 - 3(\boldsymbol{B}\cdot\boldsymbol{x})\Phi + 2\boldsymbol{B}^2\Phi^2}. \tag{26}$$

3 Physical interpretation

In this section, we shall give some comments and physical interpretations to our solution. First, our result is obtained without any approximations and the expansion (24) is consistent with the result obtained in [9]. The behavior of the Higgs field Φ (22) against the worldsheet coordinate (z, ρ) is depicted in Fig. 2 (A). Note that in the right hand side of eq. (8) we do not persist in taking the positive branch of the square root, because it forces us to discard part of the solution given in Fig. 2 (A). The spike-like behavior of the Higgs field represents the D-string attached to the D3-brane in the brane interpretation of [11, 12]. This D-string tilts due to the uniform magnetic field and the tilt angle is exactly the one expected from the force balance [4]. Here we find that the Higgs field is multi-valued due to this tilt (see Fig. 2 (B) which shows the multi-valuedness of Φ on the $\rho = 0$ plane). This multi-valuedness is a consequence of the fact that the eq. (22) determining Φ is a fourth order algebraic equation which in general has four solutions. Another solution not depicted in Fig. 2 (B) is a fake one with $\Phi < 0$. This multi-valuedness implies that the Dirac monopole might be ill-defined as a field theoretic soliton in the non-linear BPS equation and probably also in the non-commutative BPS equation via the Seiberg-Witten map. However, the multi-valuedness is inevitable from the string theory viewpoint.



(A) (B)

Figure 2: The Higgs field of the Dirac monopole is depicted against the worldsheet coordinate (z, ρ) in the left figure (A). The right figure (B) is the one restricted to the $\rho = 0$ plane. As seen from (B) the Higgs field is multi-valued for a sufficiently large z.

Though we do not know the non-linear BPS equation for the non-Abelian case due to the complexity of the ordering in the determinant, it is expected that the Higgs field related to the non-commutative monopole by the Seiberg-Witten map is that obtained by rotating the solution of the linear BPS equation in the target space. Note that in this case of the 't Hooft-Polyakov monopole, the problematic multi-valuedness in the Dirac monopole does not necessarily appear. From the behavior near the origin $r = 0$ of the exact solution in [18, 19] with $C = \langle \Phi \rangle$,

$$\Phi = (Cr/\tanh Cr - 1)/r \sim C^2 r/3, \tag{27}$$

we can read off that the tangent vector of the deformed D3-brane is $\vec{v} = (1, -C^2/3 + B)$ and that of the worldsheet parameterization is $\vec{w} = (1, B)$ in the rotated coordinate system depicted in Fig. 3. Therefore the single-valuedness condition is expressed as the positivity of the inner product of these two vectors:

$$\vec{v} \cdot \vec{w} = 1 - C^2 B/3 + B^2 > 0. \tag{28}$$

This implies that at some value of NS 2-form even the 't Hooft-Polyakov monopole is not single-valued, which is something we have never experienced in the usual field-theoretical solitons.

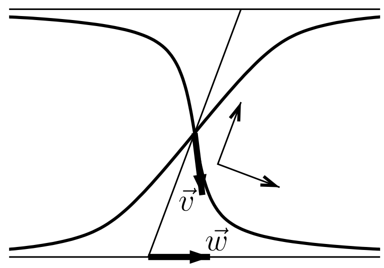


Figure 3: The 't Hooft-Polyakov monopole is not multi-valued when the single-valued condition (28) holds.

Finally, we would like to comment on the small B expansion (24). Unlike our experience of the non-commutative monopole in the flat space [5, 7] where the only parameter is θ, at $O(B^2)$ of (24) we find terms proportional to r^{-1} as well as r^{-5}, which implies the parameter $2\pi\alpha'$ also appears. If we expect the present result is transformed to the non-commutative monopole by the Seiberg-Witten map, this kind of double expansion seems impossible.

The resolution to this paradox is given by considering the moduli of the open string metric G and the non-commutativity parameter θ. When we relate the non-commutative gauge theory to its commutative counterpart, we should also relate the moduli by [3]

$$\frac{1}{G} + \theta = \frac{1}{g+B}. \tag{29}$$

Since we set the metric in the commutative space to the flat one $g_{ij} = \delta_{ij}$ and turn on only the spatial NS 2-form B_i, our moduli are

$$G_{ij} = (1 + \boldsymbol{B}^2)\delta_{ij} - B_i B_j, \qquad \theta^{ij} = -\frac{\epsilon_{ijk} B_k}{1 + \boldsymbol{B}^2}, \tag{30}$$

with a necessarily non-trivial open string metric. Using these open string moduli we can construct several kinds of scalars:

$$R^2 \equiv G_{ij} x^i x^j = (1 + B^2)\rho^2 + z^2, \tag{31}$$

$$\theta \cdot x \equiv \sqrt{G}\epsilon_{ijk}\theta^{jk} x^i = -Bz, \tag{32}$$

$$\theta^2 \equiv G^{ij}\sqrt{G}\epsilon_{ikl}\theta^{kl}\sqrt{G}\epsilon_{jmn}\theta^{mn} = B^2. \tag{33}$$

Note that in the case of a non-trivial metric the ϵ tensor should always be accompanied with $\sqrt{G}$. In terms of these scalars our result (22) can be rewritten into an expression with only one parameter θ:

$$\left(R^2 + 2\theta \cdot x\Phi + \theta^2\Phi^2\right)\Phi^2 = q^2. \tag{34}$$

Note that from the dimensional ground there can appear no $2\pi\alpha'$ in (34). Our observation here shows that we have two viewpoints for the non-linear monopole. One is with the flat space and a NS 2-form and the other is with the non-trivial metric and the non-commutativity parameter. Similarly if we rewrite the magnetic field (26) into the covariant field strength, we will also find an expression without $2\pi\alpha'$:

$$F_{ij} = \frac{\sqrt{G}\epsilon_{ijk}(-x^k - 2\Phi\theta^k)\Phi}{R^2 + 3\theta \cdot x\Phi + 2\theta^2\Phi^2}. \tag{35}$$

Similar expression for the gauge field is difficult to find because of the existence of the Dirac string.

Our analysis so far is believed to be related to the non-commutative monopole by the Seiberg-Witten map [3, 8, 9]. This kind of transformation is possible owing to the fact that the solution is unchanged even when the derivative corrections to the Born-Infeld action are taken into account [20]. In the case of a constant non-trivial metric the BPS equation in the non-commutative space should be given by

$$\widehat{F}_{ij} = \sqrt{G}\epsilon_{ijm}G^{mn}\widehat{D}_n\widehat{\Phi}, \tag{36}$$

as can be seen from the BPS bound arguments [18, 19]. However since the non-trivial metric is constant, we can always orthonormalize it globally by the vielbein:

$$E^i_\alpha E^j_\beta G_{ij} = \delta_{\alpha\beta}. \tag{37}$$

Therefore if we would like to find the non-commutative monopole in the flat space we have to collect all our results of the non-linear BPS equation, rewrite them in the covariant form, make a coordinate transformation $x^i \to E^\alpha_i x^i$ into the flat space, and transform them into the non-commutative space by the Seiberg-Witten map. Our result in the flat space up to $O(\theta^2)$ for the Higgs field is

$$\Phi = \frac{q}{r} - \frac{q^2\theta \cdot x}{r^4} - \frac{q^3\theta^2}{2r^5} + \frac{5q^3(\theta \cdot x)^2}{2r^7}, \tag{38}$$

where we have rewritten R into r because now we are in the flat space. And the gauge field corresponding to the field strength (35) is given by

$$A_i = A^0_i + A^1_i, \tag{39}$$

with

$$A_1^0 = \frac{qy}{r(r+z)}, \quad A_2^0 = -\frac{qx}{r(r+z)}, \quad A_3^0 = 0, \tag{40}$$

$$A_i^1 = \frac{q^2\epsilon_{ijk}\theta_j x_k}{r^4} - \frac{5q^3\epsilon_{ijk}\theta_j x_k \theta_m x_m}{2r^7}. \tag{41}$$

Note that due to the presence of the Dirac string the solution in the zero-th order in θ cannot be written in a spherically symmetric form. We have explicitly transformed this result into the non-commutative space by the Seiberg-Witten map [21]

$$\widehat{A}_i = A_i - \frac{1}{2}\theta^{kl}A_k(\partial_l A_i + F_{li}) + \frac{1}{2}\theta^{kl}\theta^{mn}A_k(\partial_l A_m \partial_n A_i - \partial_l F_{mi} A_n + F_{lm}F_{ni}), \tag{42}$$

$$\widehat{\Phi} = \Phi - \frac{1}{2}\theta^{kl}A_k(2\partial_l\Phi) + \frac{1}{2}\theta^{kl}\theta^{mn}A_k(\partial_l A_m \partial_n \Phi - \partial_l\partial_m\Phi A_n + F_{lm}\partial_n\Phi), \tag{43}$$

and checked that it indeed satisfies the non-commutative BPS equation by using a symbolic manipulation software. However due to the Dirac string the covariant form is not available and the result is too complicated and not suitable to be written here.

4 Summary and further directions

In this paper we extended the earlier idea of rotating the system to solve the non-linear BPS equation without any approximation. Since we solved it exactly, the multi-valuedness problem appeared. We also pointed out the open string metric is in general non-trivial and a careful treatment is necessary. Finally we transformed our result into the non-commutative space by the Seiberg-Witten map and confirmed it satisfies the non-commutative BPS equation.

In our exact manipulation we clarified the physical meaning of the Higgs field in the non-linear BPS equation. Hence in the non-Abelian case, even though we do not know the non-linear BPS equation, we expect the solution for the Higgs field related to the non-commutative monopole is that obtained by rotating the solution of the linear BPS equation in the target space. However the meaning of the gauge field is still unclear. So we do not know what to expect for the gauge field. To understand it is an interesting subject.

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